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The diffraction accompanying the regular reflexion of a plane obliquely impinging shock wave from the walls of an obtuse wedge

By S. M. TER-MINASSIANTS
Computing Centre, Moscow University

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The diffraction problem of a plane shock wave at the apex of an obtuse wedge treated by Lighthill (1950) is extended by assuming that the shock wave strikes the walls of the obtuse wedge at some finite oblique angle of incidence (not exceeding the critical angle). Transformations similar to that performed in the above-mentioned paper lead to a non-symmetrical boundary-value problem for an analytic function of a complex variable having a non-homogeneity in the form of a delta-function. It was found possible to extend, for the case considered, the method developed by Lighthill and construct the solution in almost as simple a form as given in the above-mentioned paper. The case of three-dimensional stationary flow is considered when the line of reflexion makes a finite angle with the edge of the wedge.

1. Introduction

A plane shock wave striking the rigid wall at a finite angle of incidence α and reflecting regularly from it, reaches, at the moment $t = 0$, a point where the slope of the wall changes abruptly by a small angle ϵ (positive if this abrupt change forms an obtuse wedge of angle $2\pi - \epsilon$) and continues its motion leaving behind the reflected shock front, which undergoes a slight distortion, an adjoining region of diffraction.

In what follows it will be assumed that the condition of regular reflexion is always satisfied. It relates each value of the strength of the impinging wave to a highest possible value of the angle of incidence, which decreases from the value $\frac{1}{2}\pi$, corresponding to an extremely weak acoustic wave, to a minimum value α_* equal for air to 39° 14′, after which it increases slightly when the strength of the wave increases indefinitely. A detailed treatment of the regular reflexion may be found in chapter V of Von Mises' (1958) monograph.

Further, it is assumed that the angle of incidence does not lie in the immediate vicinity of its critical value; for in this case, in the vicinity of the point of reflexion (point N in figure 1), even in the case of a weak incident wave, the flow is non-uniform and the magnitude of the pressure exceeds the value given by the theory of regular reflexion. The effects which arise in this case are treated in detail in the investigation of Rijov & Christianovitch (1958).

The angle ϵ is the small parameter of the problem. The investigated motion is considered as a small perturbation of the uniform flow of the gas, moving with a constant velocity behind the plane reflected shock front, which makes a finite angle γ with the wall. Accordingly in the equations governing the plane non-stationary motion of the gas and in the boundary conditions only the first terms of the expansions of the unknown functions—the pressure, density and velocity components—in powers of that parameter are retained, the boundary conditions on the slightly distorted reflected shock front being satisfied on the plane front corresponding to its undisturbed state.

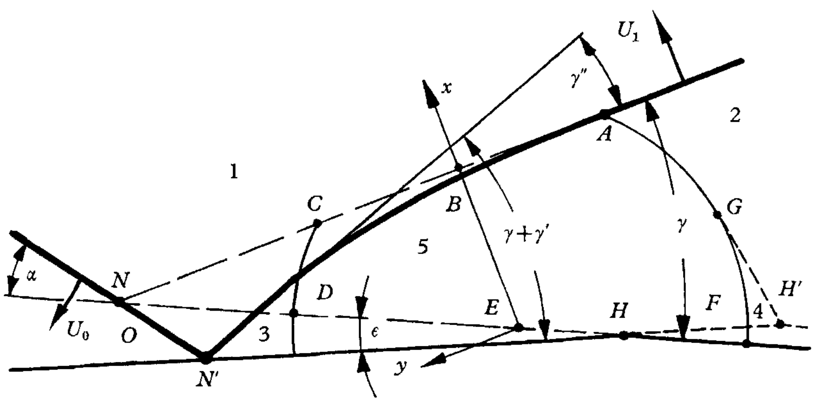

Figure 1

Therefore, it is assumed that the region of diffraction is limited by the portion ABC of the reflected front (see figure 1), the portion DEF of the wall and by the arcs CD and AF of the Mach circle (with its centre coinciding with the gas particle which at the moment $t = 0$ was at the point of reflexion N, and its radius equal to the velocity of sound in the gas through which the incident and undisturbed reflected fronts have travelled).

Depending on the strength of the incident wave and on the magnitude of the angle of incidence, the point where the slope of the wall changes abruptly may be either in (the point H) or out (the point H') of the diffraction region; the corresponding cases are called subsonic and supersonic. In figure 1 the supersonic case is represented by a dashed line; it may occur only in the case of strong shock waves and great angles of incidence.

In the subsonic case the region of diffraction is adjoined along the arc AF by a uniform undisturbed gas flow, caused by the incident and reflected waves before the point of reflexion has passed the corner and along the arc CD by another uniform flow in the vicinity of the point of reflexion after it has passed the corner.

In both cases the first of these flows is considered as basic, the pressure, density and velocity disturbances being zero there.

In the supersonic case in the vicinity of the point of intersection F of the arc FA and the wall, the region of diffraction is adjoined by a uniform flow with non-vanishing constant disturbances, caused by the supersonic flow passed the small inclination of the wall. The other portion of the boundary of this region is formed by the Mach line, passing through the point H' and tangent to the arc AF at the point G.

In the absence, among the determining parameters, of a characteristic linear dimension the problem is self-similar; for describing the resulting flow, instead of two special co-ordinates and time, we can thus introduce two non-dimensional independent variables.

2. The regular reflexion of a shock wave from a wall

The necessary formulae derived in this section are either taken from Von Mises' (1958) monograph or may be easily deduced from the relation contained in it.

It is convenient to number the regions of uniform flows described above as shown in figure 1 and designate the pressure p, density ρ and velocity of sound a relating to these regions with the corresponding indices.

Given the strength and the angle of incidence of the impinging wave, the angle of reflexion is defined as the smaller (in absolute value) root of the quadratic equation

$$C_1 \tan^2\gamma + C_2 \tan\gamma + C_3 = 0 \tag{2.1}$$

with coefficients

$$\left.\begin{aligned} C_1 &= \frac{\kappa+1}{\kappa-1}(\lambda \operatorname{cosec}^2\alpha - 1), \\ C_2 &= \frac{2}{\kappa-1}(\lambda^2 \operatorname{cosec}^2\alpha - 1)\cot\alpha, \\ C_3 &= \frac{2}{\kappa-1}\lambda\left(\lambda - \frac{3-\kappa}{2}\right)\operatorname{cosec}^2\alpha - 1. \end{aligned}\right\} \tag{2.2}$$

The quantity λ in these relations is expressed in terms of the ratio of pressures p_1/p_0 or in terms of velocity U of the impinging wave in still air ($M_0 = U_0/a_0$, κ is the specific heat ratio) as follows:

$$\lambda = \left(\frac{\kappa+1}{\kappa-1}\frac{p_1}{p_0}+1\right)\Big/\frac{2}{\kappa-1}\left(\frac{p_1}{p_0}-1\right) = \frac{\kappa+1}{2}\frac{M_0^2}{M_0^2-1}. \tag{2.3}$$

The changes in the parameters of the gas across the incident shock are easily obtained from conservation laws and may be written in the form

$$\begin{aligned} p_1 &= \frac{2\rho_0}{\kappa+1}\left(U_0^2 - \frac{\kappa-1}{2\kappa}a_0^2\right), \quad \rho_1 = \frac{\kappa+1}{\kappa-1}\rho_0\Big/\left(1+\frac{2}{\kappa-1}\frac{a_0^2}{U_0^2}\right), \\ V_1 &= \frac{2}{\kappa+1}U_0\left(1-\frac{a_0^2}{U_0^2}\right). \end{aligned} \tag{2.4}$$

The velocity of the gas W_1 in the region 1 relative to the point of reflexion in terms of the velocity V_1 of the gas in this region relative to the wall may be obtained by projecting these two velocities on the perpendicular to the wall

$$W_1 = \frac{2}{\kappa+1}U_0\left(1-\frac{a_0^2}{U_0^2}\right)\frac{\cos\alpha}{\sin\beta}. \tag{2.5}$$

The flow rotation angle in this expression (the same for incident and reflected waves) is defined by the well-known formula

$$\cot\beta = \tan\alpha\left[\frac{\kappa+1}{\alpha}M_0\operatorname{cosec}^2\alpha\Big/(M_0^2-1)-1\right]. \tag{2.6}$$

The velocity of propagation U_1 of the reflected wave, relative to the gas ahead of it, is equal to the projection on the normal to its front of the velocity vector W_1

$$U_1 = W_1\sin(\beta+\gamma). \tag{2.7}$$

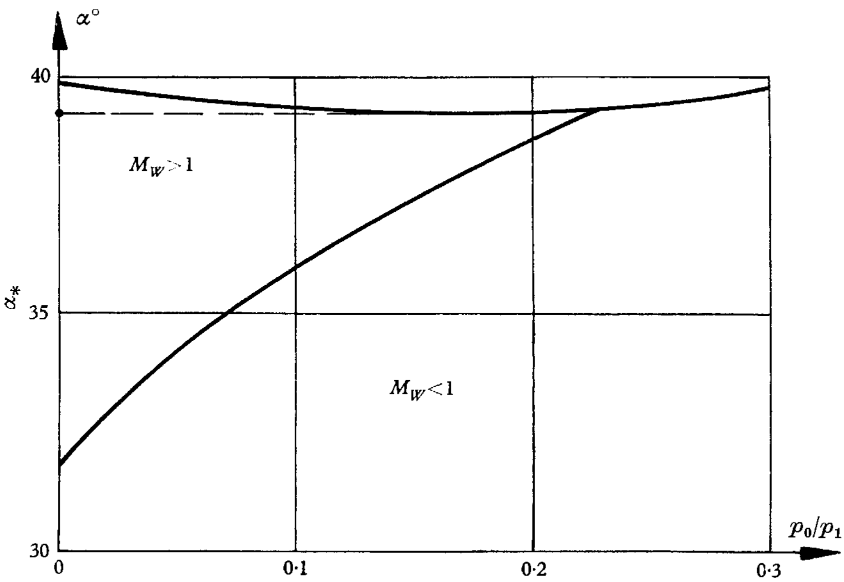

FIGURE 2

To obtain the magnitudes of the velocities of the gas, V_2 and W_2, behind the reflected shock wave relative to the gas ahead of its front and to the point of reflexion and the magnitudes of the pressure and density p_2 and ρ_2 in this region, it is obvious that one should only substitute the quantities U_1, a_1 and $\beta+\gamma$ instead of U_0, and α in the formulae (2.4) and (2.5).

At this point one should mention the role of the minimum value of the critical angle of incidence: for arbitrary strengths of the incident wave, $\gamma = \alpha_*$ and $p_2(\alpha_*)/p_0 = p_2(0)/p_0$ when $\alpha = \alpha_*$. Differentiating the solution of (2.1) with respect to α one may state that, for $\alpha = 0$ and for a value of α in the interval $0 < \alpha < \alpha_*$, $d[p_2(\alpha)/p_0]/d\alpha = 0$. Hence the variation of $p_2(\alpha)/p_0$ when α varies by ϵ in the vicinities of these points will be of order ϵ^2, i.e. will vanish in the case of the linear approximation considered. For the vicinity of the point $\alpha = 0$, this was noted by Lighthill (1950). If ϵ is not too small, this may occur on the whole interval $0 < \alpha < \alpha_*$; then on both sides of the diffraction region the pressures are equal (except the vicinity of the point F in the supersonic case). However, for $\alpha > \alpha_*$ or $\alpha > \alpha_*$ in the case of very small ϵ the difference in these pressures will be of order ϵ; therefore in what follows the formulae are derived independently of the facts noted herein. It is obvious that the subsonic or supersonic case will occur depending on whether the quantity

$$M_W = M_0(a_0/a_2)\operatorname{cosec}\alpha - M \tag{2.8}$$

($M_W = V_W/\alpha_2$, $M = W_2/a_2$), characterizing the velocity of the gas behind the reflected shock relative to the wall, is less or greater than unity.

The values of the strengths and angles of incidence of the incident waves are plotted in figure 2 for the supersonic cases. These values are located in the region of high strengths and adjoin the portion of the curve of highest possible values of the angles of incidence in the case of regular reflexion including the point $\alpha = \alpha_*$, plotted also in figure 1. The remaining values of the strengths and angles of incidence of the incident waves represented by the points below the two curves plotted in figure 2 correspond to subsonic cases.

In the subsonic case the quantity M_W denotes the distance from the centre E of the Mach circle, limiting the region of diffraction, to the point H where the slope of the wall changes abruptly (see figure 1); in the supersonic case M_W denotes the distance from the same centre to the point H', and thus defines the position of the point of tangency G of this circle with the Mach line from the point H' in terms of the angle θ'_G between the radius EG and the portion EH' of the wall

$$\sec\theta'_G = M_W \quad \text{when} \quad M_W > 1. \tag{2.9}$$

It may easily be shown by simple calculation that if the diffraction occurs in air ($\kappa = 1\cdot 4$), the point G moves from the point F to the point A with increasing strength and angle of incidence but never reaches it when the reflexion is regular.

3. Introduction of the small parameter

For simplicity of treatment of the self-similar motion considered, let us divert our attention from the whole reflexion pattern of an obliquely impinging shock wave from the wall, assuming the reflected shock as a plane shock wave propagating in still air in region 1 normally to its front. It is convenient to connect the system of non-dimensional co-ordinates (x, y) with the gas behind the reflected shock wave, its origin coinciding with the gas particle which in the moment $t = 0$ was at the point of reflexion. The x-axis is directed perpendicular and the y-axis parallel to the undisturbed reflected shock front.

The co-ordinates (x, y) are given by the formulae

$$x = (X - V_2 t)/a_2 t, \quad y = Y/a_2 t, \tag{3.1}$$

where (X, Y) are the co-ordinates connected with the gas ahead of the reflected shock front.

When the expressions for the pressure, density and velocity components in region 2 added to small disturbances of these values,

$$\bar{p} = p_2 + p', \quad \bar{\rho} = \rho_2 + \rho', \quad \bar{u} = V_2 + u', \quad \bar{v} = v',$$

are substituted in the system of equations governing the plane non-stationary motion of an ideal gas:

$$\left.\begin{aligned}\frac{\partial\bar{\rho}}{\partial t}+\nabla.(\bar{\rho}\mathbf{W})=0,\quad \frac{\partial\mathbf{W}}{\partial t}+(\mathbf{W}.\nabla)\mathbf{W}=-\frac{1}{\bar{\rho}}\nabla\bar{p},\\ \left(\frac{\partial}{\partial t}+\mathbf{W}.\nabla\right)(\bar{p}\bar{\rho}^{-\kappa})=0\end{aligned}\right\}\tag{3.2}$$

(the bar over the letters denotes total quantities in any region), new non-dimensional unknown functions are introduced,

$$p=p'/\rho_2a_2V_2,\quad \rho=\rho'/\rho_2,\quad u=u'/V_2,\quad v=v'/V_2,\tag{3.3}$$

and using the last equation the density is eliminated it reduces to a system of three linear equations

$$x\frac{\partial p}{\partial x}+y\frac{\partial p}{\partial y}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y},\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\partial p}{\partial x},\quad x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{\partial p}{\partial y}.\tag{3.4}$$

In (x, y) co-ordinates the equation of the undisturbed reflected wave front is

$$x=\frac{U_1-V_2}{a_2}=\frac{W_2\sin\gamma}{a_2}=M\sin\gamma=m.\tag{3.5}$$

In order to obtain the boundary conditions on the curved portion of the reflected shock front the value of U_0 in the expressions (2.4) must be replaced by the normal component of the local velocity of the shock front. The necessary transformations are carried out in Lighthill's (1949) paper and therefore will not be given here (the only complication is that κ may assume an arbitrary constant value).

If the equation of the curved portion of the reflected shock front is written in the form

$$x=m+f(y),\tag{3.6}$$

where $f(y)$ is assumed to be a small quantity of order ϵ, then for the unknown functions, at points on that portion adjacent to the region of diffraction, we obtain the expressions ($M_1=U_1/a_1$)

$$\left.\begin{aligned}u&=\frac{a_2}{U_1}\frac{M_1^2+1}{M_1^2-1}[f(y)-yf'(y)],\\ v&=-f'(y),\\ p&=\frac{p_2}{V_2\rho_2}\frac{2U_1}{U_1^2-\dfrac{\kappa-1}{2\kappa}a_1^2}[f(y)-yf'(y)],\end{aligned}\right\}\tag{3.7}$$

which, consistently with the formulation of the problem, are assumed to be satisfied on the straight line $x=m$. From (3.7), by dividing and differentiating on this straight line, the following conditions are obtained

$$u=Ap,\quad y\,\partial v/\partial y=B\,\partial p/\partial y,\tag{3.8}$$

where $$A=\frac{M_1^2+1}{2M_1^2}\left[\frac{2\kappa M_1^2-\kappa+1}{2+(\kappa-1)M_1^2}\right]^{\frac{1}{2}},\quad B=\frac{\kappa+1}{2}\frac{M_1^2-1}{2+(\kappa-1)M_1^2}.\tag{3.9}$$

The value of p in the region 3 may be obtained from the third equation (3.7) when the equation of the rectilinear portion of the reflected shock front limiting this region is known. Its slope to the axis y may be determined if in expressions (2.1) and (2.8) α is replaced by $\alpha+\epsilon$, γ by $\gamma+\gamma'$ and only terms of first degree in ϵ and γ' are retained; simple but cumbersome transformations lead to the result

$$\gamma'/\epsilon = (\sin 2\gamma/2\tan\alpha) \times \frac{\lambda^2(2\tan\alpha-3\tan\gamma)+[(1-\lambda^2)\tan\alpha\tan\gamma+(\kappa+1)\tan^2\gamma-(3-\kappa)\lambda]\tan\alpha}{\lambda[\lambda(\tan\gamma-2\tan\alpha)+(3-\kappa)\tan\alpha]-[\tan\gamma-(\kappa-1)\tan\alpha]\sin^2\alpha}. \tag{3.10}$$

The quantity $\gamma+\gamma'$ is the angle of the reflected shock front on the left side (see figure 1) and the quantity γ the angle on the right side of the obtuse wedge; therefore the required angle will be

$$\gamma'' = \gamma'+\epsilon = f'. \tag{3.11}$$

The equation of the unknown portion of the reflected shock front is written as the equation of a straight line passing through the point of contact N' (see figure 1) of the incident wave NN' and the wall of the wedge HN. The straight line NN' is defined by its slope to the axis equal to $\alpha+\gamma$ and by the distance $NB = M\cos\gamma$, the straight line HN' by the slope to the y-axis, equal to $\gamma-\epsilon$, and by the distance $EH = M_W$.

The resulting expression for the quantity p in the region 3 is (to within the assumed accuracy)

$$p = \frac{2p_2}{V_2\rho_2}\frac{U_1\cos\gamma}{U_1^2-\dfrac{\kappa-1}{2\kappa}a_2^2}\left(\gamma''M-\epsilon M_0\frac{a_0}{a_2}\frac{1+\tan\gamma\cot\alpha}{\sin\alpha}\right). \tag{3.12}$$

Note that the expressions (3.10) and (3.12) together with all the expressions entering into the solution of the problem are of the same order of accuracy and are derived on the assumption that ϵ is small compared with α. However, if in some specific case the quantity α is of the same order as ϵ (although ϵ may be sufficiently small) the formulae (3.10) and (3.12) will be inaccurate. In these and also in all other cases the quantity γ', and depending on it the constant quantities, may be obtained by repeating the calculations using the non-linear relations in §2. When α is small we may use Lighthill's solution.

The easiest way to obtain the conditions on the wall and on the Mach arc FA is to rotate the co-ordinate axes (x, y) until the axis y coincides with the normal to the right portion of the wall (see figure 1). Let these co-ordinates and the corresponding velocity components be denoted by $(\tilde{x}, \tilde{y})$ and $(\tilde{u}, \tilde{v})$ and let the equations (3.4) be written in this system of co-ordinates. It is evident that on the portion DH in the subsonic case and on the portion DF in the supersonic case $v = \epsilon$; on the portion HF in the subsonic case $v = 0$. From the last of equations (3.4) it then follows that on the portion DF of the wall we have

$$\partial p/\partial\tilde{y} = -\epsilon M_W\delta(\tilde{x}-\tilde{x}_H) \quad \text{when} \quad M_W \leqslant 1, \quad \partial p/\partial\tilde{y} = 0 \quad \text{when} \quad M_W > 1 \tag{3.13}$$

since on this portion $\tilde{y} = 0$ and $\partial\tilde{v}/\partial x = -\epsilon\delta(\tilde{x}-x_H)$, where δ is a delta-function of $\tilde{x}$, $\tilde{x}_H = M_W$.

The quantity p in the region 4 is defined by the supersonic flow past a small angle

$$p = -\epsilon M_W/(M_W^2-1)^{\frac{1}{2}}. \tag{3.14}$$

Along the Mach arc FA, $p = 0$ in the subsonic case; in the supersonic case $p = 0$ on AG and is defined by (3.14) on GF. The position of the point G is given by the formula (2.9). Thus on the arc FA we have

$$\left.\begin{aligned} p = \partial p/\partial\theta' &= 0 \quad \text{when} \quad M_W \leqslant 1, \\ \partial p/\partial\theta' &= -\epsilon M_W \delta(\theta'-\theta'_G)/(M_W^2-1)^{\frac{1}{2}} \quad \text{when} \quad M_W > 1, \end{aligned}\right\} \tag{3.15}$$

where $\delta(\theta'-\theta'_G)$ is a delta-function of θ', the polar angle in the co-ordinate system $(\tilde{x}, \tilde{y})$.

4. Conversion to harmonic functions

It is well known that the system of equations (3.4), after eliminating the functions u and v, passing to polar co-ordinates (r, θ)

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{4.1}$$

and performing the transformation

$$r = 2R/(1+R^2), \tag{4.2}$$

on the radii vectors, becomes the Laplace equation for the function p.

The only portion of the boundary of the diffraction region which deforms by the transformation (4.2) is the chord line ABC, $r = m/\cos\theta$, which becomes a circle (see figure 3),

$$2R\cos\theta = m(1+R^2) \tag{4.3}$$

cutting the unit circle $R = r = 1$ orthogonally at $\theta = \pm\cos^{-1}m$ as pointed out by Lighthill (1949).

In that same paper it is shown that the boundary conditions (3.8) on the reflected shock front may be reduced to a condition for only the function p, which, after the transformation (4.2), is satisfied on the circle arc (4.3). If n and s are the co-ordinates along the outward normal and tangent to the contour of the region of diffraction in the system of co-ordinates (R, θ) (the positive direction of s corresponds to describing the contour anticlockwise) then this condition has the form

$$\frac{\partial p/\partial n}{\partial p/\partial s} = \frac{Am\tan\theta - B\cot\theta}{(1-m^2\sec^2\theta)^{\frac{1}{2}}}. \tag{4.4}$$

The transformation (4.2) changes likewise the condition (3.13) at the corner in the subsonic case. Taking into account the relation $\delta[r(R)] = \delta(R)/|dr/dR|$, it will have the form

$$\partial p/\partial n = -\epsilon M_W \delta(R-R_H)/(1-M_W^2)^{\frac{1}{2}}, \tag{4.5}$$

where

$$R_H = [1-(1-M_W^2)^{\frac{1}{2}}]/M_W; \tag{4.6}$$

the remaining boundary conditions are not changed by the transformation (4.2), one should only note the relation

$$\theta_G = \theta'_G - \tfrac{1}{2}\pi - \gamma. \tag{4.7}$$

5. Formulation of the Hilbert problem

The region of diffraction in the plane $\zeta = R\exp(i\theta)$ (see figure 3) is a curvilinear quadrangle with orthogonally intersecting arcs of circles and segments of straight lines. A bilinear transformation transferring the points of intersection 1 and 2 of the perpendicular, dropped from the centre O of the circle ABC to the diameter DEF, with the unit circle respectively to the points O and ∞, transforms this quadrangle into a concentric semi-annulus, since the above mentioned circle and the diameter become arcs of circles which are orthogonal to two straight lines, the transforms of the straight line 1–2 and of the unit circle. The resulting semi-annulus is mapped conformally into a rectangle by the logarithmic function. The superposition of these two conformal mappings defines the function

$$\left.\begin{aligned} z &= \log\frac{\zeta - \exp(i\theta_2)}{\zeta - \exp(i\theta_1)} - i\frac{\theta_2 - \theta_1}{2},\\ \theta_1 &= \sin^{-1}M^{-1} - \gamma, \quad \theta_2 = \pi - \sin^{-1}M^{-1} - \gamma, \end{aligned}\right\} \tag{5.1}$$

and the rectangle is the region of diffraction in the plane

$$z = \sigma + i\tau \quad \text{(see figure 3)}. \tag{5.2}$$

The right vertical side of this rectangle $\sigma = l$ corresponds to the reflected shock, the left vertical side $\sigma = 0$ to the wall, $0 < \tau < \pi$; the arcs CD and FA correspond to the horizontal sides $\tau = \pi$ and $\tau = 0$, $0 < \sigma < l$ respectively.

The positions of the point G on the side FA and of the point H on the portion EF of the side DEF are determined from (2.8), (2.9), (4.6), (4.7) and (5.1) respectively for the super- and subsonic cases

$$\sigma_G = \tfrac{1}{2}\log\frac{MM_W + 1 + [(M^2-1)(M_W^2-1)]^{\frac{1}{2}}}{MM_W + 1 - [(M^2-1)(M_W^2-1)]^{\frac{1}{2}}}, \quad \tau_H = \cos^{-1}\frac{1 + MM_W}{M + M_W}. \tag{5.3}$$

In order to obtain the boundary condition on the transformed reflected shock front it is sufficient to consider only the transformation of the right-hand side of the relation (4.4) by the conformal mapping (5.1). The inversion of the transformation (5.1) for $\sigma = l$ gives

$$\tan\theta = \frac{m_0}{M\tan\gamma}\frac{m_0 - M\cos\tau}{M - m_0\cos\tau}, \quad m_0 = [1 - (M^2-1)\tan^2\gamma]^{\frac{1}{2}}; \tag{5.4}$$

substituting it in (4.4) leads to the condition on ABC

$$\frac{\partial p/\partial s}{\partial p/\partial n} = b(\tau) = \frac{P}{Q} = \frac{m_0^2 M(M^2-1)^{\frac{1}{2}}\tan\gamma(m_0 - M\cos\tau)\sin\tau}{mm_0^2 A(m_0 - M\cos\tau)^2 - M^2B\tan^2\gamma(M - m_0\cos\tau)^2}. \tag{5.5}$$

The condition on the transform of the wall DEF, according to (4.5) and (4.6), may be written in the form

$$\left.\begin{aligned}\partial p/\partial\sigma &= -\epsilon M_W \delta(\tau-\tau_H)/(1-M_W^2)^{\frac{1}{2}} \quad \text{when} \quad M_W \leqslant 1,\\ \partial p/\partial\sigma &= 0 \quad \text{when} \quad M_W > 1.\end{aligned}\right\} \tag{5.6}$$

The condition on the transform of the Mach arc FA, according to (3.15), takes the form

$$\left.\begin{aligned}\partial p/\partial\sigma &= 0 \quad \text{when} \quad M_W \leqslant 1,\\ \partial p/\partial\sigma &= -\epsilon M_W \delta(\sigma-\sigma_G)/(M_W^2-1)^{\frac{1}{2}} \quad \text{when} \quad M_W > 1,\end{aligned}\right\} \tag{5.7}$$

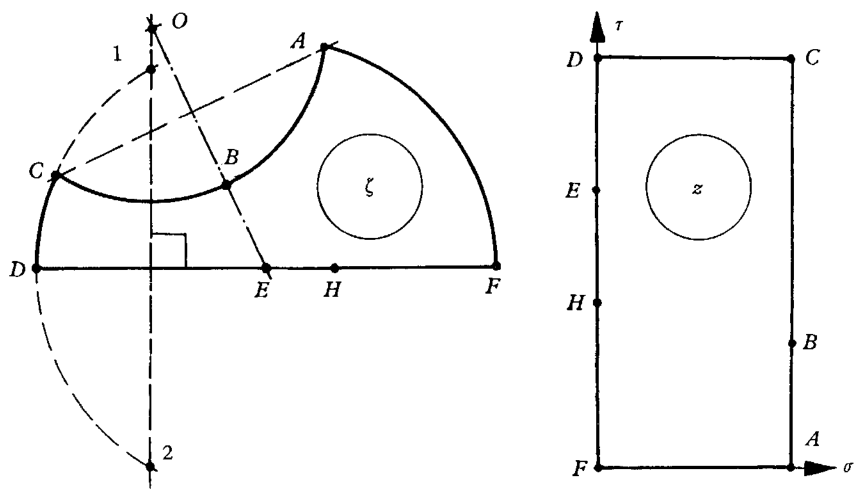

FIGURE 3

and finally that on the transform of the Mach arc CD takes the form

$$\partial p/\partial\sigma = 0. \tag{5.8}$$

The whole system of boundary conditions for the derivatives in the z-plane may be written as a single relation

$$P\,\partial p/\partial\sigma - Q\,\partial p/\partial\tau = S, \tag{5.9}$$

if it is assumed that according to (5.5) $P/Q = b(\tau)$, $S = 0$ on ABC, that on $CDEFA$ $P = 1$, $Q = 0$ and S is given by the right-hand sides of the expressions (5.6), (5.7) and (5.8) on the corresponding elements of the contour.

The relation (5.9) is the boundary condition for the non-homogeneous Hilbert problem for a function, analytic in the rectangle (5.2),

$$\Gamma(z) = \partial p/\partial\sigma - i\,\partial p/\partial\tau. \tag{5.10}$$

The unknown derivatives of the function p must satisfy also two integral conditions. The first of these, obtained by integration of the second relation (3.8) on the reflected wave front and indicated by Lighthill (1950) in his investigation of the symmetrical case of this problem, defines the normalization of the solution for the pressure. The second condition naturally arises when passing to the non-symmetrical case and does not need further clarification. In the variables σ, τ these conditions are written in the form

$$B\int_0^{\pi}\frac{\partial p}{\partial \tau}\frac{d\tau}{y(\tau)} = -\gamma'', \quad \int_0^{\pi}\frac{\partial p}{\partial \tau}d\tau = \frac{p_3-p_2}{\rho_2 a_2 V_2}. \tag{5.11}$$

According to the theory of boundary-value problems for analytic functions (see Muskhelishvili 1953) the determination of the solution of the formulated problem requires first finding the solution of the corresponding homogeneous problem

$$\Gamma_0(z) = \partial p^0/\partial\sigma - i\,\partial p^0/\partial\tau, \tag{5.12}$$

which is obtained by assuming that the right-hand side S in the condition (5.9) vanishes everywhere on the contour.

The coefficients in the boundary condition of this problem are discontinuous at the points A and C; therefore in order to construct the solution it is necessary to know its character in the vicinity of these points. In what follows, we shall determine a solution, which is continuous in the closed region (5.2) and, therefore, bounded at these points.

6. Lighthill's method in the absence of symmetry

According to the theory of boundary-value problems we may write explicitly the expression for the function $\Gamma_0(z)$ in terms of a Cauchy type integral. However, the procedure employed by Lighthill (1950) in the symmetrical case leads in a simple way to expressions which are more effective in carrying out calculations. The representation of the argument of the unknown analytic function on the transform of the reflected wave front in the form of two additive terms of the type $\tan^{-1}(\alpha\tan\tau)$ permits easy calculation (by means of residue theory) of the integrals which determine its Fourier coefficients and an extension of every term of the series into the domain, after multiplying it by the required elliptic function, to find the solution.

It turns out that in the non-symmetrical case, in spite of the relative complexity of the expression (5.5), the argument of $\Gamma_0(z)$ on the transform of the reflected shock front also admits a simple representation of the indicated type, that is on ABC (see figure 3) we have

$$\arg\Gamma_0(z) = \tan^{-1}b(\tau) = \sum_{j=1}^{4}\tan^{-1}(E_j\tan\tfrac{1}{2}\tau), \tag{6.1}$$

where

$$E_{1,2,3,4} = \left[\frac{M+m_0}{M-m_0}\right]^{\frac{1}{2}}[D_{1,2}\pm(D_{1,2}^2-1)^{\frac{1}{2}}],$$

$$D_{1,2} = \frac{1-m^2\pm\{1-m^2-4mB[(1-m^2)A-mB]\}^{\frac{1}{2}}}{2[(1-m^2)A-mB]}. \tag{6.2}$$

The expression (6.1) indicates that $\arg\Gamma_0(z)$ increases by 2π along the transform ABC of the reflected wave front when moving from point A to point C. From (5.9) it follows that on CD and FA, $\partial p^0/\partial\sigma = \operatorname{Re}\Gamma_0(z) = 0$. On the other hand, the denominator of the expression (5.5) tends to a finite limit when $\tau\to 0$ and $\tau\to\pi$, whereas the numerator, and therefore $\partial p/\partial\tau = \operatorname{Im}\Gamma_0(z)$, in both cases tends to zero as a linear function. By virtue of the assumed properties of the function $\Gamma_0(z)$, it follows that the points A and C are its simple zeroes and that, if the limiting value of $\arg\Gamma_0(z)$ when $\tau\to 0$, $\tau>0$, $\sigma=l$ (at the point A) is chosen as its zero value then, since A and C are corner points of the contour, $\arg\Gamma_0(z)$ equals $3\pi/2$ on CD and $\frac{1}{2}\pi$ on FA.

In the case of an antisymmetric boundary condition on ABC, the position of the third simple zero of the function $\Gamma_0(z)$ is readily fixed in the middle of the left vertical side of the rectangle (5.2), since at this point $\partial p^0/\partial\sigma=0$ according to the boundary condition on DEF and $\partial p^0/\partial\tau=0$ in virtue of symmetry of the function p. This zero conditions a jump in the value of $\arg\Gamma_0(z)$ by π and separates the portions DE and EF of this side where the values of $\arg\Gamma_0(z)$ are equal to its above-mentioned values on the adjoining horizontal sides CD and FA. Thus, in this case, when describing the closed contour, $\arg\Gamma_0(z)$ resumes the zero value.

Without the symmetry property the values of $\arg\Gamma_0(z)$ on ABC are otherwise distributed, but its total increase on this portion of the contour, as is readily seen from (6.1), remains the same. The solution for $\Gamma_0(z)$ for the case of non-antisymmetric argument will coincide with Lighthill's solution when $\alpha\to 0$ if the total variation of $\arg\Gamma_0(z)$, when describing the contour, will also vanish. Thus, it should be assumed that on loss of the symmetry property the simple zero of the function $\Gamma_0(z)$ is displaced from the middle of the left vertical side of the rectangle (5.2) along the contour. The position of this zero $z=z_0$ in general is not known *a priori*; however, with the availability in this connexion of the second condition (5.11) the whole system of boundary conditions is closed.

The solution $\Gamma_0(z)$ of the homogeneous Hilbert problem, bounded at points of discontinuity of the coefficients in the boundary condition and whose argument does not vary when describing the contour, is unique.

The function $\Gamma_0(z)$ is represented in the form of a product

$$\Gamma_0(z)=c\Lambda(z)L(z) \tag{6.3}$$

of functions $\Lambda(z)$ and $L(z)$ analytic in (5.2), the first of which, if the constant 2π be subtracted from its argument, satisfies the boundary condition on ABC and has vanishing argument on the remaining portions of the contour, while $\arg L(z)$ is equal to 2π on ABC, to $\frac{3}{2}\pi$ on the portion from the point C to some beforehand unknown point of the contour z_0 (where $\Gamma_0(z_0)=0$) and to $\frac{1}{2}\pi$ on the portion from this point to the point A. The quantity C in the expression (6.3) is so far an unknown constant.

In order to determine the function $\Lambda(z)$ we must write out the Fourier sine series in the interval $0<\tau<\pi$, of the function (6.1) reduced by the constant 2π, as

$$-\sum_{n=0}^{\infty}\left(4-\sum_{j=1}^{4}F_j^n\right)n^{-1}\sin n\tau,\quad F_j=(E_j-1)/(E_j+1). \tag{6.4}$$

It is easily seen that this expression will be the imaginary part of the logarithm of the function $\Lambda(z)$ with $z=l+i\tau$, $0<\tau<\pi$ if we put

$$\Lambda(z)=\exp\left[-\sum_{n=1}^{\infty}\left(4-\sum_{j=1}^{4}F_j^n\right)n^{-1}\operatorname{cosech} nl\cosh nz\right]. \tag{6.5}$$

It is obvious that on $CDFA$ $\arg\Lambda(z)=\operatorname{Im}\log\Lambda(z)=0$.

It may be shown that the series in the square brackets in (6.5) converges at any point of the region and its contour and uniformly converges to an analytic function in any closed region consisting of internal and boundary points except the points on the right vertical side of the rectangle (5.2).

As in the symmetric case the function $\Lambda(z)$ may be represented by infinite products on isolated portions of the contour. On the transform of the wall, i.e. on the left vertical side DF of the rectangle (5.2), the last factor in the square brackets takes the form $\operatorname{cosec} nl \cos n\tau$, which permits us to use the series

$$\operatorname{cosech} nl = 2q^{\frac{1}{2}n}/(1-q^n) = 2(q^{\frac{1}{2}n}+q^{\frac{3}{2}n}+q^{\frac{5}{2}n}+\dots);$$

substituting it in (6.5), changing the order of summation and taking into account the identity

$$-2\sum_{n=1}^{\infty} F^n n^{-1} \cos\tau = \log(1-2F\cos\tau+F^2)$$

leads to the expression

$$\Lambda^-(i\tau) = \prod_{n=0}^{\infty} \frac{[1-2q^{n+\frac{1}{2}}\cos\tau+q^{2n+1}]^4}{\prod_{j=1}^{4}[1-2q^{n+\frac{1}{2}}F_j\cos\tau+q^{2n+1}F_j^2]}. \tag{6.6}$$

The symbol Λ^- denotes the contour value of the function Λ.

The corresponding expression for the points of the transform of the Mach arc, i.e. of the side FA, differs from (6.6) only by that in it $\cos\tau$ should be replaced by $\cosh\sigma$. On the transform of the reflected shock front, i.e. on ABC, we obtain the expression

$$|\Lambda^-(l+i\tau)| = \prod_{n=0}^{\infty}{}' \frac{(1-2q^n\cos\tau+q^{2n})^4}{\prod_{j=1}^{4}(1-2q^nF_j\cos\tau+q^{2n}F_j^2)}. \tag{6.7}$$

Here prime signifies that in the case of the term $n=0$ the square root should be taken.

In order to determine the function $L(z)$ it is convenient to carry out conformal mapping of the rectangle (5.2) into the lower half-plane. It is realized by the function

$$\omega = \xi+i\eta = -k^{\frac{1}{2}}\vartheta_2(-iz,q)/\vartheta_3(-iz,q) \tag{6.8}$$

so that the points A and C become $\xi = \mp 1$, $\eta = 0$, while the points D and F become the co-ordinates $\xi = \pm\kappa$, $\eta = 0$; the co-ordinates of the point $z = z_0$, where $\Gamma_0(z_0) = 0$, become $\xi = \xi_0(z_0)$, $\eta = 0$ in the ω-plane. The quantities

$$\vartheta_1(-iz,q),\dots,\vartheta_4(-iz,q)$$

appearing in (6.8) and in the following are elliptic theta-functions (see Whittaker & Watson 1927, chapter 21); the quantity k depends on the quantity q

$$k^2 = 1-k'^2, \quad \sqrt{k'} = (1-2q+2q^4-2q^9+\dots)/(1+2q+2q^4+2q^9+\dots).$$

The construction of the function with piecewise constant arguments along the real axis is obvious; it is convenient to write it in the form

$$L(z) = L_0(z).L_1(z).L_2(z) = [\omega(z)-\xi_0(z)].\frac{-i(k'/k)^{\frac{1}{2}}}{[1-\omega^2(z)]^{\frac{1}{2}}}.\frac{1-\omega(z)}{1+\omega(z)}; \qquad (6.9)$$

the function $\omega(z)$ denotes the expression (6.8).

In the above expression the corresponding factors are separated by multiplication signs (by points). The product of the first two generalizes the similar function in Lighthill's solution, while the argument of the last function is equal to π on the transform of the shock front and vanishes on the remaining portions of the contour. In the limiting (when $\alpha \to 0$) symmetrical case, the Fourier coefficients contain terms which are absent in Lighthill's solution: the series generated by them converges to the minus logarithm of that last factor, introduced for simplifying the expressions of the Fourier coefficients when $\alpha \neq 0$.

The contour values $L^-(z)$ of the function $L(z)$ and its constituent factors on different sides of the rectangle (5.2) and the values of $L(z)$ at an arbitrary point in it are obtained from (6.9) by substituting there the contour values $\omega^- = \xi$ of the function ω (6.8), which on the sides DF, FA and AC are respectively of the form

$$\xi(i\tau) = -k^{\frac{1}{2}}\frac{\vartheta_2(\tau,q)}{\vartheta_3(\tau,q)}, \quad \xi(\sigma) = -k^{\frac{1}{2}}\frac{\vartheta_4(\sigma,q')}{\vartheta_3(\sigma,q')}, \quad \xi(l+i\tau) = -k^{\frac{1}{2}}\frac{\vartheta_3(\tau,q)}{\vartheta_2(\tau,q)}, \qquad (6.10)$$

where $\log q . \log q' = \pi^2$ or correspondingly the function (6.8).

The result of the substitution of the functions L_0 and L_2 is obvious; for the function L_1, on these same elements of the contour, we obtain the formulae

$$L_1^-(i\tau) = -ik^{\frac{1}{2}}\frac{\vartheta_3(\tau,q)}{\vartheta_4(\tau,q)}, \quad L_1^-(\sigma) = -ik^{\frac{1}{2}}\frac{\vartheta_3(\sigma,q')}{\vartheta_2(\sigma,q')}, \quad L_1^-(l+i\tau) = k^{\frac{1}{2}}\frac{\vartheta_2(\tau,q)}{\vartheta_1(\tau,q)}, \qquad (6.11)$$

at an arbitrary point in the region (5.2) we have

$$L_1(z) = -ik^{\frac{1}{2}}\vartheta_3(-iz,q)/\vartheta_4(-iz,q). \qquad (6.12)$$

7. Solution of the problem

The solution of the non-homogeneous Hilbert problem (5.5)–(5.10), after the conformal mapping (6.8) of the domain (5.2) into the lower half-plane is carried out, is represented, according to the theory of boundary-value problems for analytic functions (see Muskhelishvili 1953), by a Cauchy-type integral. The integrand contains the right-hand side S of the boundary condition (5.9) with the delta-function as a factor; this allows us to write out the solution without quadrature. After returning to the complex z-plane and introducing the designation $\Phi(z) = \Lambda(z)L_1(z)L_2(z)$, the solution in the subsonic case is represented in the form

$$\Gamma(z) = \Phi(z)\left[\frac{\epsilon M_W}{i\pi(1-M_W^2)^{\frac{1}{2}}}\frac{\xi'_\tau(i\tau_H)}{\Phi^-(i\tau_H)}\frac{1}{\xi(i\tau_H)-\omega(z)}+cL_0(z)\right] \qquad (7.1)$$

and in the supersonic case in the form

$$\Gamma(z) = \Phi(z)\left[\frac{\epsilon M_W}{i\pi(M_W^2-1)^{\frac{1}{2}}}\frac{\xi'_\sigma(\sigma_G)}{\Phi^-(\sigma_G)}\frac{1}{\xi(\sigma_G)-\omega(z)}+cL_0(z)\right]. \qquad (7.2)$$

The derivatives appearing in these formulae are given by the expressions (see Whittaker & Watson 1927, chapter 21)

$$\xi'_\tau(i\tau) = \frac{2K}{\pi k^{\frac{1}{2}}}\frac{\vartheta_1(\tau,q)}{\vartheta_4(\tau,q)}\left[\frac{1}{k}-\frac{\vartheta_2^2(\tau,q)}{\vartheta_3^2(\tau,q)}\right], \quad \xi'_\sigma(\sigma) = \frac{2K}{\pi k^{\frac{1}{2}}}\frac{k'}{k}\frac{\vartheta_1(\sigma,q')\vartheta_2(\sigma,q')}{\vartheta_3^2(\sigma,q')}, \tag{7.3}$$

where $(2K/\pi)^{\frac{1}{2}} = 1+2q+2q^4+2q^9+\ldots$; the factors constituting the function $\Phi^-(i\tau)$, $\Phi^-(\sigma)$ and the functions $\xi(i\tau)$ and $\xi(\sigma)$ are defined by the formulae (6.10) and (6.11).

The contour values of the functions (7.1) and (7.2) may be obtained, if we mentally turn again to the ω-plane, use the formula

$$\lim_{\eta\to 0} 1/(\xi_{H,G}-\xi-i\eta) = 1/(\xi_{H,G}-\xi)-i\pi\delta(\xi_{H,G}-\xi)$$

and return to the z-plane (dividing $\delta(\xi_{H,G}-\xi)$ by the modulus of the derivative of the mapping function along the contour). At points on the contour of the rectangle (5.2) the solution, in the subsonic case, has the form

$$\Gamma^-(z) = \Phi^-(z)\left[\frac{\epsilon M_W}{i\pi(1-M_W^2)^{\frac{1}{2}}}\frac{\xi'_\tau(i\tau_H)}{\Phi^-(i\tau_H)}\frac{1}{\xi(i\tau_H)-\xi(z)}+cL_0(z)\right]-\frac{\epsilon M_W}{(1-M_W^2)^{\frac{1}{2}}}\delta(\tau_H-\tau), \tag{7.4}$$

while in the supersonic case it has the form

$$\Gamma^-(z) = \Phi^-(z)\left[\frac{\epsilon M_W}{i\pi(M_W^2-1)^{\frac{1}{2}}}\frac{\xi'_\sigma(\sigma_G)}{\Phi^-(\sigma_G)}\frac{1}{\xi(\sigma_G)-\xi(z)}+cL_0(z)\right]-\frac{\epsilon M_W}{(M_W^2-1)^{\frac{1}{2}}}\delta(\sigma_G-\sigma). \tag{7.5}$$

The determination of the solution is completed by defining the constants c and $\xi_0(z_0)$ according to normalization conditions (5.11). In the subsonic case, the expression for the derivative $\partial p/\partial\tau$ along the transform of the reflected shock front appearing in these conditions, according to (5.10) and (7.4), is

$$\frac{\partial p}{\partial \tau} = -\operatorname{Im}\Gamma^-(z) = \operatorname{Im}\Phi^-(z)\left\{c[\omega(l+i\tau)-\xi_0(z_0)]-\frac{c_0}{\xi(i\tau_H)-\xi(l+i\tau)}\right\}, \tag{7.6}$$

where

$$\operatorname{Im}\Phi(z) = [\operatorname{Im}\Lambda(l+i\tau)]L_1(l+i\tau)L_2(l+i\tau), \quad \operatorname{Im}\Lambda(l+i\tau) = \frac{|\Lambda(l+i\tau)|b(\tau)}{[b^2(\tau)+1]^{\frac{1}{2}}}, \tag{7.7}$$

$$c_0 = \epsilon M_W\xi'_\tau(i\tau_H)\vartheta_4(\tau_H,q)/\pi k^{\frac{1}{2}}(1-M_W^2)^{\frac{1}{2}}\Lambda^-(i\tau_H)L_2^-(i\tau_H)\vartheta_3(\tau_H,q); \tag{7.8}$$

the quantity $b(\tau)$ is given by (5.5) or (6.1). The quantity $y(\tau) = m\tan\theta(\tau)$ appearing in the first condition (5.11) is determined by means of (3.5) and (5.4). All the functions in (7.6)–(7.8) are real; their expressions are given in §6.

Substituting (7.6) in (5.11) and taking out the unknown constants from the signs of integrals, we may easily obtain the following system of equations, linear relative to $\xi_0(z_0)$ and $1/c$,

$$\left.\begin{aligned} ck^{\frac{1}{2}}\xi_0(z_0)I_2+cI_1+c_0I_6 &= c_1,\\ ck^{\frac{1}{2}}\xi_0(z_0)I_4+cI_3+c_0I_5 &= c_2.\end{aligned}\right\} \tag{7.9}$$

The integrals appearing in (7.9),

$$\left.\begin{aligned} I_1 &= (1-m^2)^{\frac{1}{2}}\int_0^{\pi}\vartheta_3(\tau,q)\Psi(\tau)\,d\tau/y(\tau), & I_2 &= (1-m^2)^{\frac{1}{2}}\int_0^{\pi}\vartheta_2(\tau,q)\Psi(\tau)\,d\tau/y(\tau),\\ I_3 &= \int_0^{\pi}\vartheta_3(\tau,q)\Psi(\tau)\,d\tau, & I_4 &= \int_0^{\pi}\vartheta_2(\tau,q)\Psi(\tau)\,d\tau,\\ I_5 &= \int_0^{\pi}\frac{\vartheta_2(\tau,q)\Psi(\tau)\,d\tau}{\xi(i\tau_H)-\xi(l+i\tau)}, & I_6 &= \int_0^{\pi}\frac{\vartheta_2(\tau,q)\Psi(\tau)}{\xi(i\tau_H)-\xi(l+i\tau)}\frac{d\tau}{y(\tau)}, \end{aligned}\right\} \quad (7.10)$$

are calculated numerically. The expression for $\Psi(\tau)$ in the integrands is

$$\Psi(\tau) = |\Lambda^-(l+i\tau)|\,L_2^-(l+i\tau)\,b(\tau)/[b^2(\tau)+1]^{\frac{1}{2}}\vartheta_1(\tau,q). \quad (7.11)$$

The expressions for the constants c_1 and c_2 are

$$c_1 = -\gamma''(1-m^2)^{\frac{1}{2}}/B, \quad c_2 = (p_3-p_2)/\rho_2\alpha_2 V_2; \quad (7.12)$$

the quantities γ'' and B are determined respectively by (3.10), (3.11) and (3.9), while the constant c_2 is the difference of the values of p in the regions 3 and 2 (see figure 1) and is given by the right-hand side of (3.12).

The solution of the system (7.9) determines the unknown constants:

$$c = [c_1 I_4 - c_2 I_2 - c_0(I_6 I_4 - I_5 I_2)]/(I_1 I_4 - I_2 I_3), \quad (7.13)$$

$$\xi_0(z_0) = -\frac{1}{k^{\frac{1}{2}}}\frac{c_1 I_3 - c_2 I_1 - c_0(I_6 I_3 - I_5 I_1)}{c_1 I_4 - c_2 I_2 - c_0(I_6 I_4 - I_5 I_2)}. \quad (7.14)$$

It may be easily understood that in the supersonic case the same formulae (7.13) and (7.14) hold for the constants c and $\xi_0(z_0)$; however, the quantity c_0 is defined now by the expression

$$c_0 = \epsilon M_W \xi'_\sigma(\sigma_G)\vartheta_2(\sigma_G,q')/\pi k^{\frac{1}{2}}(M_W^2-1)^{\frac{1}{2}}\Lambda^-(\sigma_G)L_2^-(\sigma_G)\vartheta_3(\sigma_G,q') \quad (7.15)$$

while in the denominators of the integrands in the integrals I_5 and I_6 (7.10) the quantity $\xi(\sigma_G)$ will appear instead of $\xi(i\tau_H)$.

Once the quantity $\xi_0(z_0)$ is known, the position of the point z_0 on the contour may be determined by means of (6.10) and the tables for the theta-functions; however, there is no need of it in constructing the solution. It is completed by substituting the quantity $\xi_0(z_0)$ and c in the right-hand side of (7.1), (7.2), (7.4) and (7.5).

8. Pressure distribution along the wall

In the subsonic case the contour values (7.4) of the solution give the following expression for the derivative $\partial p/\partial\tau$ along the transform of the wall ($z = i\tau$),

$$\frac{\partial p}{\partial \tau} = -\operatorname{Im}\Gamma^-(i\tau) = \Lambda^-(i\tau)L_2^-(i\tau)\frac{\vartheta_3(\tau,q)}{\vartheta_4(\tau,q)}\left[cL_0(i\tau) - \frac{k^{\frac{1}{2}}c_0}{\xi(i\tau_H)-\xi(i\tau)}\right]. \quad (8.1)$$

The expressions for the symbols $\Lambda^-(i\tau)$, $L_2^-(i\tau)$, $L_0^-(i\tau)$, $\xi(i\tau)$ are given in §6, while the values of the constants c_0, c_1 and $\xi_0(z_0)$ in L_0 in §7. The function (8.1) tends to infinity when $\tau \to \tau_H$ and stipulates a singularity of the logarithmic type for the function p. Therefore, the calculation of the non-dimensional pressure disturbance p at points τ in the intervals $0 < \tau < \tau_H$ and $\tau_H < \tau < \pi$ should be carried out respectively by the formulae

$$p = -\int_0^\tau \operatorname{Im}\Gamma^-(i\tau)\,d\tau \quad \text{or} \quad p = c_2 + \int_0^\tau \operatorname{Im}\Gamma^-(i\tau)\,d\tau, \tag{8.2}$$

where c_2 is defined by the right-hand side of (3.12) or the right-hand side of (7.12).

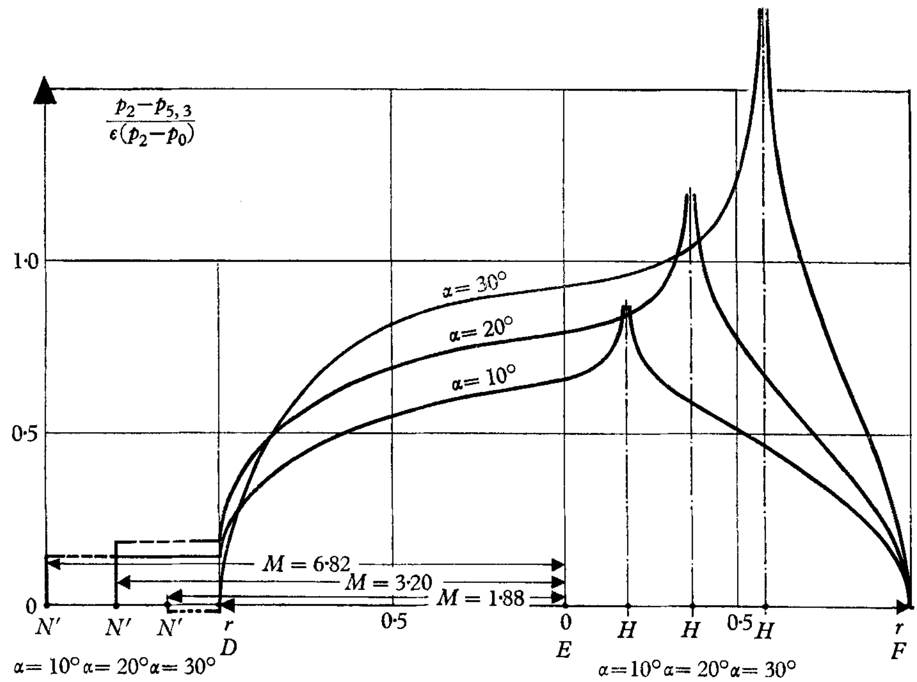

FIGURE 4

The relationship of the co-ordinate (of the radius-vector r) along the wall with the co-ordinate τ along its transform on the rectangle (5.2) obtained by back-transformation of (5.1) and (4.2), is expressed as follows:

$$r = |(M\cos\tau - 1)/(M - \cos\tau)|, \tag{8.3}$$

where r is calculated from the point E (see figure 1) in the direction of the point D when $\tau > \cos^{-1} M^{-1}$ and in the direction of the point F when $\tau < \cos^{-1} M^{-1}$.

Given the strength of the incident wave p_1/p_0 (or M_0) the angle of incidence α and the wall inclination ϵ, the calculations for the pressure distribution along the wall should be carried out in the following sequence: first the parameters γ, M, M_1, M_W, θ'_G of the regular reflexion are determined by (2.1)–(2.9); further, by means of (3.5), (3.9), (5.2), (5.3), (5.4), (6.2), (6.4) are found, respectively, the constants m, A and B, q and l, τ_H, m_0, E_j, F_j; then according to (6.6), (6.7), (6.9), (6.10), (6.11), (7.3) the functions of τ_H are determined. The calculation of the constant quantities corresponding only to the basic flow and not depending on the disturbance ϵ is carried out by computing the integrals $I_1 - I_6$ by means of

(7.10), (7.11) and (5.5), (6.7), (6.9), (6.10). The parameters which depend on ϵ are then found: γ', γ'', c_0, c_1, c_2 by (3.10), (3.11), (7.8), (7.12), (3.12). Finally, by means of (7.13), (7.14) the quantities c, $\xi_0(z_0)$ are determined; the values of the quantity $p(\tau)$ are calculated by using (8.1), (8.2) with (6.6), (6.7), (6.9), (6.10), (6.11); the corresponding points on the wall by the expression (8.3).

The curves in figure 4 illustrate the influence of the angle of incidence α (indicated in degrees by a number on each curve) on the character of the pressure distribution in the subsonic case. They correspond to the pressure ratio of the incident wave $p_1/p_0 = 3{\cdot}33$ and to an angle $\epsilon = 6°$. In all the selected cases the values of the pressure in region 3 differ slightly from its values in region 2 (see notes in §§2 and 3). The growth of the values of the quantity $(p_2-p_5)/\epsilon(p_2-p_0)$ with increasing α is distinctly seen. One may also notice the decreasing influence of the point of sudden change of the slope when $\alpha \to 0$.

In the supersonic case, the determination of the function $p(\tau)$ along the wall may be made by the use of the second formula (8.2), substituting, according to (7.5), in (8.1) the quantity $\xi(i\tau_H)$ for $\xi(\sigma_G)$ and choosing for the constant c_0 the expression (7.15).

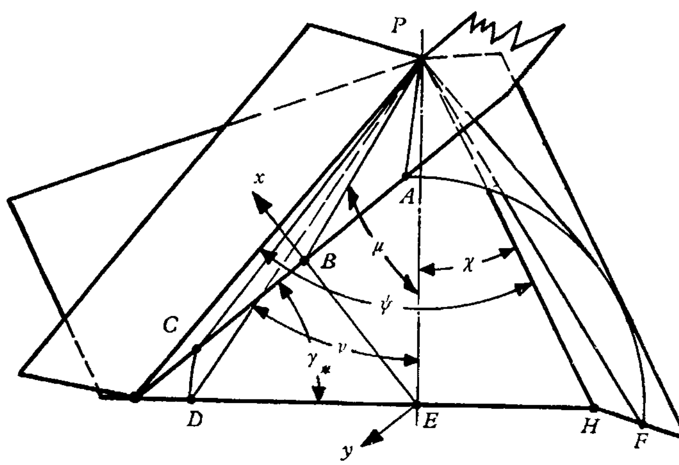

FIGURE 5

9. The three-dimensional problem

In three-dimensional co-ordinate space the diffraction studied above is represented by that particular case of a possible gas motion due to reflexion of a plane shock wave from a wall with a small sudden change of slope, when the line of reflexion is parallel to the line of sudden change of the slope of the wall. The method developed permits consideration also of the cases when these lines make a finite angle ψ.

The system of rectangular co-ordinates (x', y', z') is connected with their point of intersection P, the axis z' being directed along the vector of total velocity W_* behind the reflected shock front, the axis y' parallel to this front. The quantity W_* is the modulus of the vectorial sum of the displacement velocity, along the line of sudden change of the slope of the wall, of the point of intersection, obviously equal to $U_0/\sin\alpha\sin\psi$, and of the gas velocity V_W behind the reflected wave front relative to the wall. Then, in the part of space between the wall, the reflected wave front and the surface of the Mach cone with the apex at the point P, in the absence of a fundamental length scale, a linear stationary supersonic conical flow is realized. The detail of the flow picture is schematically represented in figure 5 (for simplicity the transfer of the boundary conditions is not shown as in figure 1, and the reflected wave front is represented as a plane wave). Chester (1954) considered a similar extension of Lighthill's two-dimensional diffraction problem of a shock wave the front of which is perpendicular to the wall before its encounter of a slight inclination of the wall. As opposed to the case analyzed by Chester when the condition $W_* > a_2$ of the existence of the conical flow restricts the magnitude of the angle between the shock front and the edge of the wedge, in the motion considered here this condition is satisfied for any angle ψ, as follows from the formula

$$M_* = W_*/a_2 = [M_0^2\operatorname{cosec}^2\alpha\cot^2\psi(a_0^2/a_2^2)+M^2]^{\frac{1}{2}}, \tag{9.1}$$

which is easily obtained from (2.8).

Having the value of M_*, we may determine the Mach cone angle ν and the angle χ between the axis z' and the line of sudden change of the slope of the wall as follows:

$$\nu = \sin^{-1}M_*^{-1}, \quad \chi = \sin^{-1}(M_W\cos\psi/M_*). \tag{9.2}$$

The introduction of non-dimensional co-ordinates

$$x = x'/z'\tan\nu, \quad y = y'/z'\tan\nu, \tag{9.3}$$

fixes the plane $z' = \cot\nu$, interesecting the Mach cone along the unit circle. The lines of intersection of this plane with the plane of the undisturbed reflected wave front and the wall form an angle

$$\gamma_* = \tan^{-1}[\tan\gamma\cos(\psi-\chi)]. \tag{9.4}$$

In the plane $z' = \cot\nu$, the distance h from the point of reflexion N to the centre E of the unit Mach circle, which plays here the same role as the quantity M in the two-dimensional problem, is

$$h = \cot\nu\,.\tan(\psi-\chi). \tag{9.5}$$

It is readily seen that the z' axis forms with the plane of the undisturbed reflected shock front an angle

$$\mu = \tan^{-1}[\tan(\psi-\chi)\sin\gamma_*]. \tag{9.6}$$

The procedure of deriving the boundary conditions on the shock front does not essentially differ from the transformations in Chester's (1954) paper and is omitted here. It is based on the establishment of a relationship between the co-ordinates (x', y', z') and the co-ordinates (X, Y, Z) connected with the gas between the incident and reflected wave fronts,

$$X = x'+V_2\cos\mu t, \quad Y = y', \quad Z = z'-(W_*+V_2\sin\mu)t, \tag{9.7}$$

and on the representation of the slightly curved reflected shock front by an equation of the form $x\tan\nu = \tan\mu + \phi(y)\sec\mu$; the following non-dimensional unknown functions are introduced: $p = p'/a_2\rho_2V_2$, $u = u'/V_2\cos\nu$, $v = v'/V_2\cos\nu$. The conditions (3.8) are obtained with coefficients (insteadof A and B)

$$\left.\begin{aligned} A_* &= \frac{1}{2}\left[\frac{2\kappa M_1^2-\kappa+1}{2+(\kappa-1)M_1^2}\right]^{\frac{1}{2}}\left[\frac{M_1^2+1}{M_1^2}-\frac{(\kappa+1)(M_1^2-1)}{(\kappa-1)M_1^2+2}\tan^2\mu\right],\\ B_* &= \frac{\kappa+1}{2}\frac{M_1^2-1}{2+(\kappa-1)M_1^2}\sec^2\mu.\end{aligned}\right\} \tag{9.8}$$

Since according to (9.6), (9.2) and (9.1) $\mu \to 0$ when $\psi \to 0$, the relations (9.7) and (9.8) become respectively (3.1) and (3.9).

In order to calculate the pressure distribution by the available (in § 8) formulae for the solution of the two-dimensional non-stationary problem, it is obvious that everywhere in these formulae the quantity ϵ should be replaced by $\epsilon_* = \epsilon\cos\psi$ (the magnitude of the angle of sudden change of the slope of the wall in the plane perpendicular to the line of reflexion); besides, instead of M_W in (2.8), use should be made of the quantity $r_H^* = \cot\nu\tan\chi$, instead of m in (3.5) the quantity $m_* = \cot\nu\tan\mu = h\sin\gamma_*$, see (9.4), (9.5), and instead of M the quantity h (9.5).

Since on the wedge wall $v = \epsilon_*\sec\nu$ the factor $\sec\nu$ should be added in (7.8) and (7.15). The indicated changes are easily established by considering figure 5 and do not require further explanation.

It is obvious that when $\psi \neq 0$ the supersonic case occurs when $\chi > \nu$ (see figure 5) which leads, according to (9.2), to the condition $M_W\cos\psi > 1$; this means that the domain of values p_1/p_0 and α for which this case takes place is obtained by contraction of the corresponding domain (see figure 2) for the two-dimensional problem; if $\kappa = 1, 4$ it disappears when $\psi > 28{\cdot}1°$.

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